Dynamics of composition operators

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1 Linear Dynamics

In linear dynamics, as the name suggests, we are interested in studying the dynamics of linear operators $T : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a separable topological vector space. In this notes, we shall consider only operators acting on Banach spaces, that is, complete normed spaces, although many of the results are valid in more general spaces (such as Fréchet or F-spaces). The results and examples of this and the next sections can be found in the excellent books [1] and [4], both of which treat linear dynamics in depth.

A very natural kind of chaotic behavior that one can think for a dynamical system is admitting a point for which the orbit is dense:

Definition 1.1. We say that $T : \mathcal{X} \to \mathcal{X}$ is *hypercyclic* if there exists $x \in \mathcal{X}$ such that the set

$$\mathrm{orb(x, T)} = \{T^n(x) : n \in \mathbb{N}\}$$

is dense in $\mathcal{X}$.

A powerful tool to show that an operator is hypercyclic is Birkhoff transitivity theorem, by showing that it is topologically transitive:

Definition 1.2. We say that $T : \mathcal{X} \to \mathcal{X}$ is *topologically transitive* if for any pair of nonempty open sets $U, V \subseteq \mathcal{X}$, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$.

Theorem 1.3 (Birkhoff transitivity theorem). *Let $T : \mathcal{X} \to \mathcal{X}$ be a continuous map on a separable complete metric space without isolated points. Then the following are equivalent:*

(1) *T is topologically transitive;*

(2) *T admits a point with dense orbit.*

If any of these conditions hold, the set of points with dense orbit is a G_δ-set.

Proof. (2) $\implies$ (1): Suppose that the orbit of $x \in \mathcal{X}$ is dense. Since $\mathcal{X}$ does not have isolated points, the orbit of $T^p(x)$ is also dense, for any integer $p \geq 0$. Now let $U, V \subseteq \mathcal{X}$ be a pair of nonempty open sets. Let $n \geq 0$ be such that $T^n(x) \in U$. As the orbit of $T^n(x)$ is dense, there exists $m \geq n$ such that $T^m(x) \in V$. Hence $T^{m-n}(U) \cap V \neq \varnothing$.

(1) $\implies$ (2): Since $\mathcal{X}$ is metrizable and separable, it admits a countable base of nonempty open sets $(U_k)_{k\geq 1}$. Let $HC(T) \subseteq \mathcal{X}$ be the set of points in $\mathcal{X}$ that have dense orbit under T. Note that $x \in HC(T)$ if, and only if, for every $k \geq 1$, there exists $n \geq 0$ such that $T^n(x) \in U_k$. That is,

$$HC(T) = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} T^{-n}(U_k).$$

By continuity of T, for every $k \geq 1$ the set $\bigcup_{n=0}^{\infty} T^{-n}(U_k)$ is open, and by the topological transitivity, it is also dense. By the Baire category theorem, the set $HC(T)$ is a G_δ-set, in particular it is nonempty. $\square$

Remark. We did not require that T is linear in the previous theorem.

The next example illustrates how we can use this theorem to show that an operator is hypercyclic.

Example 1.4 (Rolewicz's operators)**.** Let $\mathcal{X} = \ell^p$ and $T : \ell^p \to \ell^p$ be given by

$$T(x_1, x_2, x_3, \ldots) = (\lambda x_2, \lambda x_3, \lambda x_4, \ldots),$$

where $\lambda \in \mathbb{K}$. If $|\lambda| \leq 1$, for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$ we have $\|T^n(x)\| = |\lambda|^n \cdot \|(x_{n+1}, x_{n+2}, \ldots)\| \leq \|x\|$, so that T cannot be hypercyclic.

Suppose now that $|\lambda| > 1$. Let $U, V \subseteq \mathcal{X}$ be nonempty open sets. Since the set of finite sequences is dense, there exists $N \in \mathbb{N}$ and points $x, y \in \ell^p$ of the form

$$x = (x_1, \ldots, x_N, 0, 0, \ldots) \quad \text{and} \quad y = (y_1, \ldots, y_N, 0, 0, \ldots)$$

such that $x \in U$ and $y \in V$. Let $n \geq N$. Define $z \in \mathcal{X}$ as

$$z_k = \begin{cases} x_k, & 1 \leq k \leq N; \\ \lambda^{-n} y_{k-n}, & n+1 \leq k \leq n+N; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $T^n(z) = y$. We also have that $\|x - z\| = |\lambda|^{-n}\|y\| \to 0$. Hence, we have that if n is sufficiently large, $z \in U$ and $T^n(z) \in V$. This shows that T is topologically transitive, so using Birkhoff transitivity theorem we conclude that T is hypercyclic.

We can also study stronger notions of chaoticity. One of them is the notion of mixing operators:

Definition 1.5. We say that $T : \mathcal{X} \to \mathcal{X}$ is *mixing* if for any pair of nonempty open sets $U, V \subseteq \mathcal{X}$, there exists some $N_0 \geq 0$ such that $T^n(U) \cap V \neq \varnothing$ for all $n \geq N_0$.

A very useful result that enables us to show that some operators are mixing is Kitai's Criterion:

Theorem 1.6 (Kitai's Criterion)**.** *Let T be an operator on a separable Banach space $\mathcal{X}$. If there are dense subsets $\mathcal{X}_0, \mathcal{Y}_0 \subseteq \mathcal{X}$ and a map $S : \mathcal{Y}_0 \to \mathcal{Y}_0$ such that, for any $x \in \mathcal{X}_0$ and $y \in \mathcal{Y}_0$*

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy = y$,

then T is mixing.

Proof. Let $U, V \subseteq \mathcal{X}$ be nonempty open sets and take $x \in U \cap \mathcal{X}_0$ and $y \in V \cap \mathcal{Y}_0$. Let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subseteq U$ and $B(y, \varepsilon) \subseteq V$. By (i) and (ii), there exists N_0 such that, for all $n \geq N_0$,

$$\|T^n(x)\| < \varepsilon \quad \text{and} \quad \|S^n(y)\| < \varepsilon.$$

We have that, for all $n \geq N_0$,

$$\|x - (x + S^n(y))\| = \|S^n(y)\| < \varepsilon,$$

so that $x + S^n(y) \in U$. Furthermore, using (iii),

$$\|T^n(x + S^n(y)) - y\| = \|T^n(x) + T^n(S^n(y)) - y\| = \|T^n(x)\| < \varepsilon,$$

showing now that $T^n(x + S^n(y)) \in V$. Thus, $T^n(U) \cap V \neq \varnothing$ for all $n \geq N_0$. $\square$

Remark. 1) Note that we did not assume that S is linear or continuous in Kitai's Criterion.

2) It is possible to show that there exist mixing operators that do not satisfy Kitai's Criterion, see [3] and [2].

As an application of Kitai's Criterion, we show that if $|\lambda| > 1$, the associated Rolewicz's operator is mixing:

Example 1.7. Let us consider again the Rolewicz's operators: let $\lambda \in \mathbb{K}$ be such that $|\lambda| > 1$ and consider $T : \ell^p \to \ell^p$ given by

$$T(x_1, x_2, x_3, \ldots) = (\lambda x_2, \lambda x_3, \lambda x_4, \ldots).$$

Take $\mathcal{X}_0 = \mathcal{Y}_0 = c_{00}$, i.e. the space of finitely supported sequences, and let

$$S(x_1, x_2, x_3, \ldots) = (0, \lambda^{-1} x_1, \lambda^{-1} x_2, \lambda^{-1} x_3, \ldots).$$

One can easily see that conditions (i)-(iii) of Kitai's Criterion are satisfied. Hence, we have shown that T is even mixing if $|\lambda| > 1$.

2 Weighted Shifts

This section is dedicated to a very important class of operators in linear dynamics that generalizes Rolewicz's operators: the one of weighted shifts.

Let $w = (w_n)_n$ be a bounded sequence of nonzero scalars. We consider the operator $B_w : \ell^p \to \ell^p$ given by

$$B_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots).$$

Note that the boundedness of (w_n) implies that B_w is well-defined and continuous.

In order to characterize the hypercyclic weighted shifts, we will need the following theorem, which is a weaker version of Kitai's Criterion. The proof is analogous to the latter, hence we omit it.

Theorem 2.1 (Hypercyclicity Criterion)**.** *Let T be an operator on a separable Banach space $\mathcal{X}$. If there are dense subsets $\mathcal{X}_0, \mathcal{Y}_0 \subseteq \mathcal{X}$, a map $S : \mathcal{Y}_0 \to \mathcal{Y}_0$ and a strictly increasing sequence $(n_k)_{k \geq 1}$ such that, for any $x \in \mathcal{X}_0$ and $y \in \mathcal{Y}_0$*

(i) $T^{n_k} x \to 0$,

(ii) $S^{n_k} y \to 0$,

(iii) $TSy = y$,

then T is hypercyclic.

Theorem 2.2. *$B_w : \ell^p \to \ell^p$ is hypercyclic if, and only if,*

$$\sup_{n\geq 1} \prod_{\nu=1}^{n} |w_\nu| = \infty.$$

Proof. Suppose first that B_w is hypercyclic and let $x = (x_1, x_2, \dots) \in \ell^p$ be a point with dense orbit. We have that

$$B_w^n(x) = \Big(\Big(\prod_{\nu=2}^{n+1} w_\nu\Big)x_{n+1}, \Big(\prod_{\nu=3}^{n+2} w_\nu\Big)x_{n+2}, \dots\Big).$$

As the orbit of x is dense, there exists a strictly increasing subsequence $(n_k)_{k\geq 1}$ such that

$$\lim_{k\to\infty} B_w^{n_k}(x) = e_1 = (1, 0, 0, \dots).$$

Thus

$$\Big(\prod_{\nu=2}^{n_k+1} w_\nu\Big)x_{n_k+1} \to 1.$$

Since $x \in \ell^p$, we have that $x_{n_k+1} \to 0$, so that

$$\Big(\prod_{\nu=2}^{n_k+1} w_\nu\Big) \to \infty.$$

Now suppose that the condition holds. Let us show that B_w satisfies the Hypercyclicity Criterion. Let $\mathcal{X}_0 = \mathcal{Y}_0 = c_{00}$ be the dense subset of finite sequences and $S = F_w : c_{00} \to c_{00}$ be given by

$$F_w(x_1, x_2, x_3, \dots) = (0, w_2^{-1}x_1, w_3^{-1}x_2, \dots).$$

It is easy to see that conditions (i) and (iii) satisfied for the full sequence (n). For (ii), note that, for every $n \geq 0$,

$$F_w^n(e_j) = \Big(\underbrace{0, \dots 0}_{n+j-1}, \prod_{\nu=j+1}^{n+j} w_\nu^{-1}, 0, 0, \dots\Big),$$

so that

$$\|F_w^n(e_j)\| = \prod_{\nu=j+1}^{n+j} |w_\nu|^{-1}.$$

Let $(n_k)_k$ be a strictly increasing sequence such that

$$\prod_{\nu=j+1}^{n_k+j} |w_\nu|^{-1} < \frac{1}{k}$$

for every $1 \leq j \leq k$. Hence,

$$F_w^{n_k}(e_j) \to 0$$

for every $j \geq 1$, so by linearity the result follows. □

Using Kitai's Criterion instead of the Hypercyclicity Criterion, one can also characterize mixing weighted shifts:

Theorem 2.3. *$B_w : \ell^p \to \ell^p$ is mixing if, and only if,*

$$\lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty.$$

3 Composition Operators

For a more complete treatment on composition operators acting on spaces of measurable functions, we refer to [5].

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $f : X \to X$ be a measurable function. We will assume that f is a non-singular transformation, i.e. for every $B \in \mathcal{B}$, $\mu(B) = 0$ implies that $\mu(f^{-1}(B)) = 0$. In this case, the push forward measure $f_*\mu$, defined by $f_*\mu(B) = \mu(f^{-1}(B))$, for every $B \in \mathcal{B}$, is absolutely continuous with respect to μ. By the Radon-Nikodym theorem, there exists a measurable function $g : X \to [0, \infty)$ such that

$$\mu(f^{-1}(B)) = \int_B g d\mu, \qquad \forall B \in \mathcal{B}.$$

Our goal will be to study the composition operator induced by f given by

$$\begin{aligned} T_f : L^p(\mu) &\to L^p(\mu) \\ \varphi &\mapsto \varphi \circ f. \end{aligned}$$

We first note that f being non-singular implies that T_f is well-defined:

Proposition 3.1. *If f is non-singular, then the composition operator T_f is well-defined.*

Proof. Let $\varphi_1, \varphi_2 \in L^p(\mu)$ such that $\varphi_1 = \varphi_2$ μ-almost everywhere. Then, there exists a set $N \in \mathcal{B}$ such that $\mu(N) = 0$ and

$$\varphi_1(x) = \varphi_2(x) \qquad \forall x \in X \setminus N.$$

Hence $\varphi_1(f(x)) = \varphi_2(f(x))$ for all $x \in X \setminus f^{-1}(N)$. By the non-singularity of f, we have that $\mu(f^{-1}(N)) = 0$, so that $\varphi_1 \circ f = \varphi_2 \circ f$ μ-almost everywhere, showing that T_f is well-defined. $\square$

Remark. If f is not a non-singular transformation, then T_f is not necessarily well-defined. Indeed, let $X = [0, 1]$ with the Lebesgue measure and $f : X \to X$ be given by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \dfrac{1}{2}; \\ 1, & \dfrac{1}{2} < x \le 1. \end{cases}$$

Since

$$\mu(\{1\}) = 0 \qquad \text{and} \qquad \mu(f^{-1}(\{1\})) = \mu([1/2, 1]) = \frac{1}{2},$$

f is not non-singular.

Let

$$\varphi_1 = \chi_{[0,1)} \qquad \text{and} \qquad \varphi_2 = \chi_{[0,1]},$$

so that φ_1 and φ_2 represent the same element in $L^p(\mu)$, since they are equal μ-almost everywhere. However, we have that, for every $x \in [1/2, 1]$,

$$0 = \varphi_1(f(x)) \neq \varphi_2(f(x)) = 1.$$

This shows that T_f is not well-defined.

Theorem 3.2. *$T_f : L^p(\mu) \to L^p(\mu)$ is a continuous linear operator if, and only if, there exists $c > 0$ such that*

$$\mu(f^{-1}(B)) \le c\mu(B) \qquad \forall B \in \mathcal{B}. \tag{$\star$}$$

Proof. Suppose that T_f is continuous. Let $B \in \mathcal{B}$ be such that $\mu(B) < \infty$. We have that

$$\mu(f^{-1}(B)) = \int_{f^{-1}(B)} 1 d\mu = \int_X \chi_B \circ f d\mu = \|\chi_B \circ f\|^p = \|T_f \chi_B\|^p \leq \|T_f\|^p \|\chi_B\|^p = \|T_f\|^p \mu(B).$$

By taking $c = \|T_f\|^p$, the result follows. The case where $\mu(B) = \infty$ is always true.

Conversely, suppose that condition ($\star$) holds. Let

$$g = \frac{d(f_* \mu)}{d\mu}$$

be the Radon-Nikodym derivative of $f_* \mu$ with respect to μ. We claim that $g \leq c$ almost everywhere. Indeed, suppose that there exists a measurable set A such that $\mu(A) > 0$ and $g(x) > c$ for every $x \in A$. By the σ-finiteness of X, we may assume that $\mu(A) < \infty$. Then, we would have that

$$\mu(f^{-1}(A)) = \int_A g d\mu > c\mu(A),$$

contradicting our hypothesis and proving our claim.

Now let $\varphi \in L^p(\mu)$. We have

$$\|T_f \varphi\|^p = \int_X |\varphi \circ f|^p d\mu = \int_X |\varphi|^p d(f_* \mu) = \int_X |\varphi|^p g d\mu \leq c \cdot \int_X |\varphi|^p d\mu = c \cdot \|\varphi\|^p,$$

showing that T_f is continuous. □

Remark. We have that condition ($\star$) is equivalent to the boundedness of the Radon-Nikodym derivative

$$g = \frac{d(f_* \mu)}{d\mu}.$$

Indeed, by the proof of the previous theorem, we have that if ($\star$) holds, then g is bounded. Conversely, if $g \leq c$ and $B \in \mathcal{B}$, then

$$\mu(f^{-1}(B)) = \int_B 1 d(f_* \mu) = \int_B g d\mu \leq c\mu(B).$$

We say that $f^{-1}(\mathcal{B})$ *is essentially all of* $\mathcal{B}$, and denote it by $f^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$, if given $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\mu(A \Delta f^{-1}(B)) = 0$.

To characterize hypercyclic composition operators, we will need the following lemmas.

Lemma 3.3. *If $f^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$, then for every $A \in \mathcal{B}$ and $k \geq 1$, there exists $B \in \mathcal{B}$ such that*

$$\mu(f^{-k}(B) \Delta A) = 0.$$

Lemma 3.4 ([6, Lemma 1])**.** *If T_f has dense range, then $f^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$.*

Proof. Let $A \in \mathcal{B}$ be a set of finite measure. By hypothesis, there exists a sequence φ_n of functions in $L^p(\mu)$ such that $T_f \varphi_n \to \chi_A$. By passing to a subsequence if necessary, we have that $T_f \varphi_n \to \chi_A$ almost everywhere, i.e. there exists $N \in \mathcal{B}$ such that $\mu(N) = 0$ and, for all $x \in X \setminus N$,

$$T_f \varphi_n(x) = \varphi_n \circ f(x) \to \chi_A(x).$$

Since $T_f \varphi_n = \varphi_n \circ f$ is $f^{-1}(\mathcal{B})$ measurable, we have that the restriction of χ_A to $X \setminus N$ is $f^{-1}(\mathcal{B})$ measurable. Thus there exists $B \in \mathcal{B}$ such that $(X \setminus N) \cap A = f^{-1}(B)$ and the result follows by the σ-finiteness of $\mathcal{B}$. □

Theorem 3.5. *$T_f : L^p(\mu) \to L^p(\mu)$ is hypercyclic if, and only if, $f^{-1}(\mathcal{B}) =_{ess} \mathcal{B}$ and for every measurable set A of finite measure and any $\varepsilon > 0$, there exist a measurable set $B \subseteq A$ and $k \geq 1$ such that*

$$\mu(A \setminus B) < \varepsilon, \qquad \mu(f^{-k}(B)) < \varepsilon \qquad \textit{and} \qquad \mu^*(f^k(B)) < \varepsilon.$$

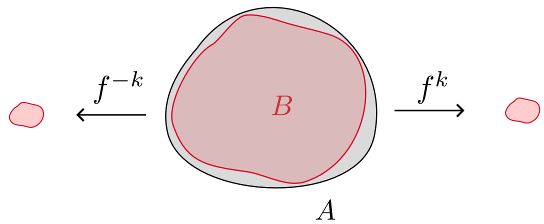

Figure 1: Hypercyclicity for composition operators

Proof. Suppose first that T_f is hypercyclic, hence topologically transitive, by the Birkhoff transitivity theorem. Since T_f has dense range, by Lemma 3.4 we get $f^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$. Let A be a measurable set of finite measure and $\varepsilon > 0$. There exist $k \geq 1$ and $\varphi \in L^p(\mu)$ such that

$$\|\varphi - 2\chi_A\| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \|\varphi \circ f^k - 4\chi_A\| < \frac{\varepsilon}{2}.$$

Since

$$\begin{aligned}\|\varphi - 2\chi_A\|^p &= \int_{\{x\in X:|\varphi(x)-2\chi_A(x)|<1\}} |\varphi - 2\chi_A|^p d\mu + \int_{\{x\in X:|\varphi(x)-2\chi_A(x)|\geq 1\}} |\varphi - 2\chi_A|^p d\mu \\ &\geq \mu(\{x \in X : |\varphi(x) - 2\chi_A(x)| \geq 1\}),\end{aligned}$$

we have

$$\mu(\{x \in X : |\varphi(x) - 2\chi_A(x)| \geq 1\}) < \frac{\varepsilon}{2}. \tag{1}$$

Analogously,

$$\mu(\{x \in X : |\varphi \circ f^k(x) - 4\chi_A(x)| \geq 1\}) < \frac{\varepsilon}{2}. \tag{2}$$

Now let

$$C = \{x \in X : |\varphi(x) - 4| < 1\} \qquad \text{and} \qquad D = \{x \in X : |\varphi(x) - 2| < 1\},$$

and define

$$B = D \cap f^{-k}(C) \cap A.$$

First, note that

$$A \setminus D \subseteq \{x \in X : |\varphi(x) - 2\chi_A(x)| \geq 1\}$$

and

$$A \setminus f^{-k}(C) \subseteq \{x \in X : |\varphi \circ f^k(x) - 4\chi_A(x)| \geq 1\},$$

so (1) and (2) imply that $\mu(A \setminus B) < \varepsilon$.

Next, we have that

$$f^{-k}(D) \subseteq \{x \in X : |\varphi \circ f^k(x) - 4\chi_A(x)| \geq 1\},$$

and hence, by (2), $\mu(f^{-k}(B)) \leq \mu(f^{-k}(D)) < \varepsilon$.

Finally,

$$C \subseteq \{x \in X : |\varphi(x) - 2\chi_A(x)| \geq 1\}.$$

Since $f^k(B) \subseteq C$, by (1) we have that $\mu^*(f^k(B)) < \varepsilon$.

Suppose now that the condition in the statement holds. Let $U, V \subseteq L^p(\mu)$ be a pair of nonempty open sets. Since the set of simple functions is dense in $L^p(\mu)$, we have that there exist a measurable set of finite measure $A \subseteq X$, simple functions $\psi_1 = \sum_{j=1}^M a_j \chi_{A_j}$, $\psi_2 = \sum_{j=1}^M b_j \chi_{B_j}$ and $\eta > 0$ such that $A_j, B_j \subseteq A$,

$$B(\psi_1, \eta) \subseteq U \quad \text{and} \quad B(\psi_2, \eta) \subseteq V.$$

Let $L = 1 + \sum_j (|a_j| + |b_j|)$.

Note that if $E, F \subseteq X$ are measurable sets of finite measure such that $\mu(E\Delta F) < (\eta/4L)^p$, then

$$\|\chi_E - \chi_F\|^p = \Big(\int_X |\chi_E - \chi_F|^p d\mu\Big)^{1/p} = (\mu(E\Delta F))^{1/p} < \eta/4L. \tag{3}$$

By the condition in the statement, there exist a measurable set $B \subseteq A$ and $k \geq 1$ such that

$$\mu(A \setminus B) < \Big(\frac{\eta}{4L}\Big)^p, \quad \mu(f^{-k}(B)) < \Big(\frac{\eta}{4L}\Big)^p \quad \text{and} \quad \mu^*(f^k(B)) < \Big(\frac{\eta}{4L}\Big)^p.$$

Let $C \in \mathcal{B}$ be such that $f^k(B) \subseteq C$ and $\mu(C) < (\eta/4L)^p$.

Let $\gamma_1 = \sum_j a_j \chi_{A_j \cap B}$. Since

$$\mu(A_j \Delta(A_j \cap B)) = \mu(A_j \setminus B) \leq \mu(A \setminus B) < \Big(\frac{\eta}{4L}\Big)^p, \tag{4}$$

we have by (3) that

$$\|\chi_{A_j} - \chi_{A_j \cap B}\| < \frac{\eta}{4L} \leq \frac{\eta}{2L}. \tag{5}$$

By the triangle inequality,

$$\|\psi_1 - \gamma_1\| < \frac{\eta}{2}. \tag{6}$$

If we define $\gamma_2 = \sum_j b_j \chi_{B_j \cap B}$, by a similar argument we get $\|\psi_2 - \gamma_2\| < \eta/2$.

Now since $f^{-1}(\mathcal{B}) =_{\mathrm{ess}} \mathcal{B}$, by Lemma 3.3, for every j there exists $D_j \in \mathcal{B}$ such that

$$\mu(f^{-k}(D_j)\Delta(B_j \cap B)) = 0.$$

Note that $f^{-k}(D_j)$ has finite measure. Let $C_j = D_j \cap C$ and define φ by:

$$\varphi = \sum_j a_j \chi_{(A_j \cap B)\setminus \bigcup_i C_i} + \sum_j b_j \chi_{C_j}. \tag{7}$$

First, let us show that $\varphi \in U$. Since we have that each $C_j \subseteq C$,

$$\mu\Big((A_j \cap B)\Delta\Big((A_j \cap B) \setminus \bigcup_i C_i\Big)\Big) = \mu\Big(A_j \cap B \cap \bigcup_i C_i\Big) \leq \mu\Big(\bigcup_i C_i\Big) \leq \mu(C) < \Big(\frac{\eta}{4L}\Big)^p$$

and

$$\mu(\varnothing \Delta C_j) = \mu(C_j) \leq \mu(C) < \Big(\frac{\eta}{4L}\Big)^p,$$

thus, by (3), we obtain

$$\|\chi_{A_j \cap B} - \chi_{(A_j \cap B)\setminus \bigcup_i C_i}\| < \frac{\eta}{4L} \leq \frac{\eta}{2L} \quad \text{and} \quad \|\chi_{C_j}\| < \frac{\eta}{4L} \leq \frac{\eta}{2L}.$$

Hence, using the triangle inequality,

$$\|\gamma_1 - \varphi\| < \frac{\eta}{2},$$

showing with (6) that $\varphi \in U$.

To see that $\varphi \circ f^k \in V$, note that, by (7),

$$\varphi \circ f^k = \sum_j a_j \chi_{f^{-k}((A_j \cap B) \setminus \bigcup_i C_i)} + \sum_j b_j \chi_{f^{-k}(C_j)}.$$

Since

$$\mu\Big(\varnothing \Delta \Big(f^{-k}\Big((A_j \cap B) \setminus \bigcup_i C_i\Big)\Big)\Big) \leq \mu(f^{-k}(B)) < \Big(\frac{\eta}{4L}\Big)^p$$

and

$$\mu(f^{-k}(C_j)\Delta(B_j \cap B)) \leq \mu(f^{-k}(D_j)\Delta(B_j \cap B)) = 0,$$

where $f^{-k}(C_j)$ has finite measure, by the same argument as done above we get that

$$\|\gamma_2 - \varphi \circ f^k\| < \frac{\eta}{2},$$

which shows that $\varphi \circ f^k \in V$. We have therefore shown that $T_f^k(U) \cap V \neq \varnothing$. $\square$

Analogously, we have a characterization for the mixing composition operators:

Theorem 3.6. *$T_f : L^p(\mu) \to L^p(\mu)$ is mixing if, and only if, $f^{-1}(\mathcal{B}) =_{ess} \mathcal{B}$ and for every measurable set A of finite measure and any $\varepsilon > 0$, there exist $k_0 \geq 1$ and a sequence of measurable sets $B_k \subseteq A$, $k \geq k_0$, such that, for all $k \geq k_0$*

$$\mu(A \setminus B_k) < \varepsilon, \quad \mu(f^{-k}(B_k)) < \varepsilon \quad \textit{and} \quad \mu^*(f^k(B_k)) < \varepsilon.$$

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