

Dense-lineability and spaceability in Linear Dynamics

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- X is a separable infinite dimensional F -space.

Definition

A subset $A \subset X$ is said to be dense-lineable if $A \cup \{0\}$ contains a dense linear subspace, and spaceable if $A \cup \{0\}$ contains an infinite-dimensional closed subspace of X

- 1 Dense Lineable Criterion for Linear Dynamics, A. Arbiato
- 2 On Spaceability within Linear Dynamics, M. Stadlbauer

- 1 Recall that a continuous linear operator $T : X \rightarrow X$ on a separable Banach space is called *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X . The set of such vectors is denoted by $\text{HC}(T)$
- 2 A continuous map T on X is said to be *recurrent* if the set of recurrent points, denoted by $\text{Rec}(T)$, is dense in X . Here, a point $x \in X$ is recurrent provided there exists a strictly increasing sequence $(\theta_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} T^{\theta_n} x = x$

Basic notions

Let (Z, d) be a metric space, and let $f: Z \rightarrow Z$ be a continuous map. We say that a pair $(p, q) \in Z \times Z$ is a *Li-Yorke pair* for f if:

$$\liminf_{n \rightarrow \infty} d(f^n(p), f^n(q)) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} d(f^n(p), f^n(q)) > 0.$$

A subset $\Gamma \subset Z$ is called a *scrambled set* for f if, for every pair of distinct points $x, y \in \Gamma$, the pair (x, y) is a Li-Yorke pair for f .

The map f is said to be *Li-Yorke chaotic* if there exists an uncountable scrambled set for f .

Definition

We say that a vector $x \in X$ is *irregular* for $T: X \rightarrow X$ if

$$\liminf_{n \rightarrow \infty} \|T_n(x)\| = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|T_n(x)\| = \infty. \quad (1)$$

$\text{HC}(T)$ is dense lineable

- ① Herrero, Bourdon, Bès, Wengenroth: $\text{HC}(T)$ is dense lineable.
- ② Bernal(1999). If X, Y are separable metrizable spaces and $(T_n : X \rightarrow Y)_{n \in \mathbb{N}}$ is densely hereditary hypercyclic (DHHC), then $\text{HC}((T_n)_n)$ is dense-lineable. Recall that (T_n) is DHHC whenever every subsequence $\text{HC}((T_{n_k})_k)$ is dense.

Definition

A continuous linear operator $T \in \mathcal{L}(X)$ is said to be *quasi-rigid* if there exists a sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n \uparrow \infty$ such that the set $\{x \in X : \lim_n T^{\theta_n} x = x\}$ is dense in X . Notably, if T is quasi-rigid, then $\text{Rec}(T)$ is dense lineable.

- 1 Grivaux et al. and M. Saavedra, along with M. Stadlbauer, showed that if X is a separable F -space and $T : X \rightarrow X$ is a continuous linear operator, then T is quasi-rigid if and only if, for each $n \in \mathbb{N}$, $\bigoplus_{i=1}^n T$ is recurrent.
- 2 Every complex separable infinite dimensional Banach space admit a recurrent operator T such that $\text{Rec}(T)$ is not dense-lineable. A. López + Q. Menet, A. Arbierto + M. Saavedra

Question

Let X be an infinite dimensional separable F -space, and let $\Lambda \subset \mathcal{L}(X)$ be a countable collection of quasi-rigid operators. Is it true

$$\bigcap_{T \in \Lambda} \text{Rec}(T)$$

is dense lineable?

- 1 Bermúdez et al. and Nilson Bernardes et al. established sufficient conditions under which the set of irregular vectors is dense-lineable.
- 2 Nilson Bernardes et al. established sufficient conditions for the existence of a dense mean irregular manifold.

Definition

Let X and Y be topological vector spaces. We say that Ψ is a *D-phenomenon* from X to Y if, for each $x \in X$, there exists a non-empty set $\Psi(x)$ consisting of open subsets of Y such that the following holds: for any $p = \sum_{j=1}^m \alpha_j x_j \in X$ with $\alpha_i \neq 0$ and for any $V \in \Psi(p)$, there exist non-empty open sets $W_i \in \Psi(x_i)$ such that

$$\sum_{j=1}^m \alpha_j W_j \subset V.$$

For the m -fold product X^m , we define

$$\Psi(x_1, \dots, x_m) := \left\{ \prod_{i=1}^m A_i \subset X^m : A_i \in \Psi(x_i) \right\}.$$

- Let X be a topological vector space (TVS). For each $x \in X$, define

$$\Psi^{\text{Rec}}(x) := \{\text{all open neighborhoods of } x \text{ in } X\}.$$

- Let X and Y be topological vector spaces. For each $x \in X$, define

$$\Psi^{\text{HC}}(x) := \{\text{all non-empty open subsets of } Y\}.$$

- Let X be a topological vector space and Y be a normed space. For each $x \in X$, define

$$\Psi^{\text{LY}}(x) := \{B(0, r), Y \setminus B[0, s] : r, s > 0\}$$

Definition

Let X, Y be two topological vector spaces, Ψ a D -phenomenon from X to Y , and $\mathcal{F}$ a proper Furstenberg family. We say that $x \in \mathcal{F}\Psi((T_n)_n)$ if, for every $V \in \Psi(x)$,

$$\{n \in \mathbb{N} : T_n(x) \in V\} \in \mathcal{F}.$$

Theorem

Let X be an infinite-dimensional separable F -space, $(Y_i)_{i \in \mathbb{N}}$ TVSSs, and a family of sequences of linear operators $((T_{i,n} : X \rightarrow Y_i)_{n \in \mathbb{N}})_{i \in \mathbb{N}}$. Suppose $(\Psi_i)_{i \in \mathbb{N}}$ is a family of D -phenomena from X to Y_i , and let $(\mathcal{F}_i)_{i \in \mathbb{N}}$ be a family of proper Furstenberg family. If, for every $m, i \in \mathbb{N}$, the set

$$\mathcal{F}_i \Psi_i((T_{i,n})_n : m)$$

contains a residual subset of X^m , then

$$\bigcap_{i \in \mathbb{N}} \mathcal{F}_i \Psi_i((T_{i,n})_n : m)$$

is dense-lineable for each $m \in \mathbb{N}$.

Corollary

Let X be an infinite-dimensional separable F -space, and let $\Lambda \subset \mathcal{L}(X)$ be a countable collection of quasi-rigid operators. Then,

$$\bigcap_{T \in \Lambda} \text{Rec}(T)$$

is dense lineable.

Proposición

Let X be an infinite-dimensional separable F -space, and let $\Lambda \subset \mathcal{L}(X)$ be a countable collection. Suppose that for each $T \in \Lambda$, there exists an operator $h_T \in \mathcal{L}(X)$ and a sequence $(\theta_{n,T})_n \uparrow \infty$ such that

$$\{x \in X : \lim_{n \rightarrow \infty} T^{\theta_{n,T}} x = h_T x\}$$

is dense in X for each $T \in \Lambda$. Then there exists a dense vector subspace E of X , and for each $T \in \Lambda$, there exists a sequence $(k_{n,T})_n \uparrow \infty$ such that

$$E \subset \{x \in X : \lim_{n \rightarrow \infty} T^{k_{n,T}} x = h_T x\}, \quad \forall T \in \Lambda.$$

The first work to establish sufficient conditions for the existence of a hypercyclic subspace was due to A. Montes-Rodríguez (1996). Subsequently, equivalences between the existence of hypercyclic subspaces and certain properties of the essential spectrum for operators satisfying the hypercyclicity criterion were investigated. In Hilbert spaces, these equivalences were demonstrated by F. León-Saavedra and A. Montes-Rodríguez (2001), and later generalized to Banach spaces by M. González, F. León-Saavedra, and A. Montes-Rodríguez (2000).

Sufficient conditions for recurrent subspace

Theorem (A. López, 2024)

Let X be a (real or complex) separable Banach space and let $T \in \mathcal{L}(X)$. Assume there exists an increasing sequence of integers $(k_n)_{n \in \mathbb{N}}$ such that:

- i) The set $D := \{x \in X : T^{k_n} x \xrightarrow{n \rightarrow \infty} x\}$ is dense in X ,
- ii) There exists a non-increasing sequence $(E_n)_{n \in \mathbb{N}}$ of infinite-dimensional closed subspaces of X such that

$$\sup_{n \in \mathbb{N}} \|T^{k_n}|_{E_n}\| < \infty.$$

Then T has a recurrent subspace. In particular, there exists an infinite-dimensional closed subspace F and a subsequence $(\ell_n)_{n \in \mathbb{N}}$ of $(k_n)_{n \in \mathbb{N}}$ such that $T^{\ell_n} x \rightarrow x$ for all $x \in F$.

Theorem (Gonzales et al., 2000)

Let X be a separable infinite dimensional complex Banach space, and let $T \in \mathcal{L}(X)$. Suppose that T satisfies the Hypercyclic criterion. Then the following conditions are equivalent:

- ❶ *T has a Hypercyclic subspace.*
- ❷ *There exists an infinite dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(\theta_n)_n$ such that $T^{\theta_n}x \xrightarrow[n \rightarrow \infty]{} 0$ for all $x \in E$.*
- ❸ *There exists an infinite-dimensional closed subspace $E \subset X$ such that and an increasing sequence of integers $(\theta_n)_n$ such that $\sup_n \|T^{\theta_n}|_E\| < \infty$*
- ❹ *the essential spectrum of T intersects the closed unit disk.*

Theorem (A. López, 2024)

Let X be a separable infinite dimensional complex Banach space and let $T \in \mathcal{L}(X)$. If T is quasi-rigid, then the following statements are equivalent:

- ❶ *T has a recurrent subspace;*
- ❷ *there exists an infinite-dimensional closed subspace $E \subset X$ and an increasing sequence of integers $(\theta_n)_n$ such that $T^{\theta_n} x \xrightarrow{n \rightarrow \infty} x$ for all $x \in E$;*
- ❸ *There exists an infinite-dimensional closed subspace $E \subset X$ such that and an increasing sequence of integers $(\theta_n)_n$ such that $\sup_n \|T^{\theta_n}|_E\| < \infty$;*
- ❹ *the essential spectrum of T intersects the closed unit disk.*

Our change of perspective relies on the set $\Omega(T) \subset \mathcal{L}(X)$, defined as the collection of all continuous linear operators $h : X \rightarrow X$ for which there exists a strictly increasing sequence of positive integers $(\omega_n)_n$ such that the set

$$\{x \in X : T^{\omega_n} x \xrightarrow{n \rightarrow \infty} h(x)\}$$

is dense in X .

Notably, when X is a separable infinite-dimensional Fréchet or Banach space, the set $\Omega(T)$ provides a simple characterization of important dynamical properties:

$$\begin{aligned} T \text{ is quasi-rigid} &\iff \text{Id} \in \Omega(T), \\ T \text{ is weakly mixing} &\iff \Omega(T) = \mathcal{L}(X). \end{aligned}$$

Structural perspective

To establish the connection with spaceability, for a continuous linear map $h : X \rightarrow X$ we define

$$R(T, h) := \{x \in X : \exists (\omega_n)_n \uparrow \infty \text{ such that } \lim_{n \rightarrow \infty} T^{\omega_n} x = h(x)\}.$$

T weakly mixing	T quasi-rigid
$\Omega(T) = \mathcal{L}(X)$	$F = \{\text{Id}\} \subset \Omega(T),$
$\bigcap_{h \in \mathcal{L}(X)} R(T, h) = \text{HC}(T) \cup \{0\}$	$\bigcap_{h \in \{\text{Id}\}} R(T, h) = \text{Rec}(T)$

Theorem

Let X be a complex separable infinite-dimensional Banach space, and let $T \in \mathcal{L}(X)$. Suppose that $\Omega(T)$ is non-empty. If there exists a strictly increasing sequence of positive integers $(\theta_n)_n$ and an infinite-dimensional closed subspace $E \subset X$ such that

$$\sup_n \|T^{\theta_n}|_E\| < \infty,$$

then for any SOT-separable subset $F \subset \Omega(T)$,

$$\bigcap_{h \in F} R(T, h)$$

is spaceable.