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# Zero-entropy dynamical systems with the gluing orbit property

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ABSTRACT

Under the assumption of the gluing orbit property, equivalent conditions to having zero topological entropy are investigated. In particular, we show that a dynamical system has the gluing orbit property and zero topological entropy if and only if it is minimal and equicontinuous.



Setting:
$X$ esp. métrico compacto
$f: X \to X$ contínua.

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# Gluing orbit property and partial hyperbolicity

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## Abstract

This article is a follow up of our recent works [7,8], and here we discuss the relation between the gluing orbit property and partial hyperbolicity. First we prove that a partially hyperbolic diffeomorphism with two saddles with different index, and such that the stable manifold of one of these saddles coincides with the strongly stable leaf does not satisfy the gluing orbit property. In particular, the examples of $C^1$-robustly transitive diffeomorphisms introduced by Mañé [20] do not satisfy the gluing orbit property. We also construct some families of partially hyperbolic skew-products satisfying the gluing orbit property and derive some estimates on their quantitative recurrence.

Setting: $M$ variedade $C^\infty$ fechada com $\dim M \geq 3$
$f: M \to M$ difeo $C^r$, $r \geq 1$.

GLUING-ORBIT PROPERTY, LOCAL STABLE/UNSTABLE SETS, AND INDUCED DYNAMICS ON HYPERSPACE.

MAYARA ANTUNES, BERNARDO CARVALHO, WELINGTON CORDEIRO, JOSÉ CUETO

ABSTRACT. We prove that local stable/unstable sets of homeomorphisms of an infinite compact metric space satisfying the gluing-orbit property always contain compact and perfect subsets of the space. As a consequence, we prove that if a positively countably expansive homeomorphism satisfies the gluing-orbit property, then the space is a single periodic orbit. We also prove that there are homeomorphisms with gluing-orbit such that its induced homeomorphism on the hyperspace of compact subsets does not have gluing-orbit, contrasting with the case of the shadowing and specification properties, proving that if the induced map has gluing-orbit, then the base map has gluing-orbit and is topologically mixing.

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# The gluing orbit property, uniform hyperbolicity and large deviations principles for semiflows

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Available online 22 January 2019

**Abstract**

We introduce a gluing orbit property, weaker than specification, for both continuous maps and flows. We prove that flows with the $C^1$-robust gluing orbit property are uniformly hyperbolic and that every uniformly hyperbolic flow satisfies the gluing orbit property. We also prove a level-1 large deviations principle and a level-2 large deviations lower bound for semiflows with the gluing orbit property. As a consequence we establish a level-1 large deviations principle for hyperbolic flows and every continuous observable, and also a level-2 large deviations lower bound. Finally, since many non-uniformly hyperbolic flows can be modeled as suspension flows we also provide criteria for such flows to satisfy uniform and non-uniform versions of the gluing orbit property.


*MSC:* 37D20; 37C20; 37C50; 60F10; 37C75

*Keywords:* Gluing orbit property; Specification; Semiflows; Hyperbolicity; Stability; Large deviations

$\boxed{\text{Propriedade de gluing orbit}}$

(Introduzida em Bomfim Varandas, Torres (2017/2019))

Ideia: Reconstrução de órbitas

- Mais fraca que a propriedade de especificação.

*2.1. The gluing orbit property*

**Definition 2.1.** We call a sequence

$$\mathscr{C} = \{(x_j, m_j)\}_{j\in\mathbb{Z}^+}$$

of ordered pairs in $X \times \mathbb{Z}^+$ an *orbit sequence*. A *gap* for an orbit sequence is a sequence

$$\mathscr{G} = \{t_j\}_{j\in\mathbb{Z}^+}$$

of positive integers. For $\varepsilon > 0$, we say that $(\mathscr{C}, \mathscr{G})$ can be $\varepsilon$-*traced* by $z \in X$ if the following *tracing property* (illustrated in Fig. 2) holds:

For every $j \in \mathbb{Z}^+$,

$$d(f^{s_j+l}(z), f^l(x_j)) \leq \varepsilon \text{ for each } l = 0, 1, \cdots, m_j - 1, \tag{1}$$

where

$$s_1 := 0 \text{ and } s_j := \sum_{i=1}^{j-1}(m_i + t_i - 1) \text{ for } j \geq 2.$$

$\forall \varepsilon > 0, \ \exists \ M = M(\varepsilon)$

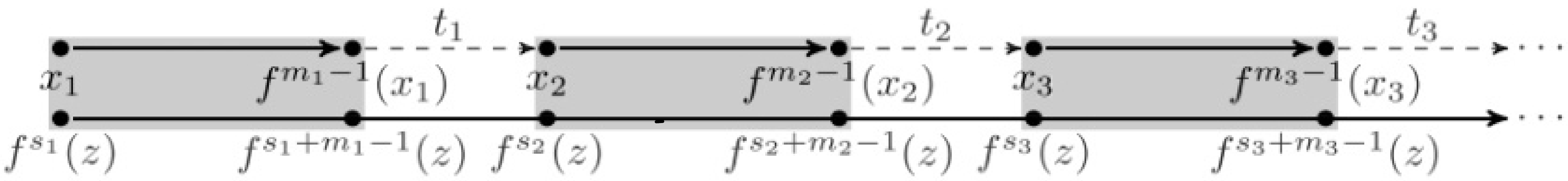
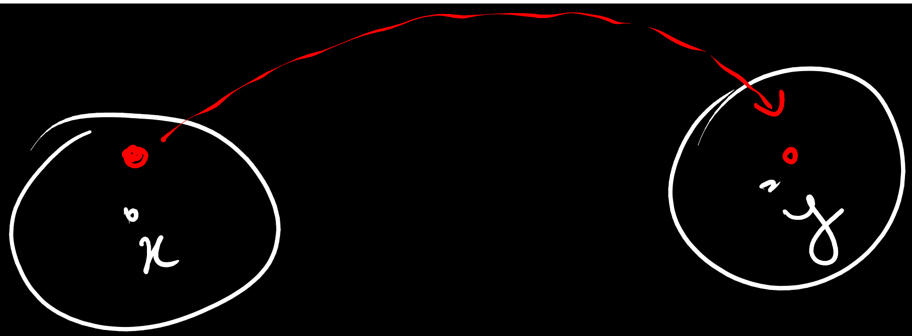


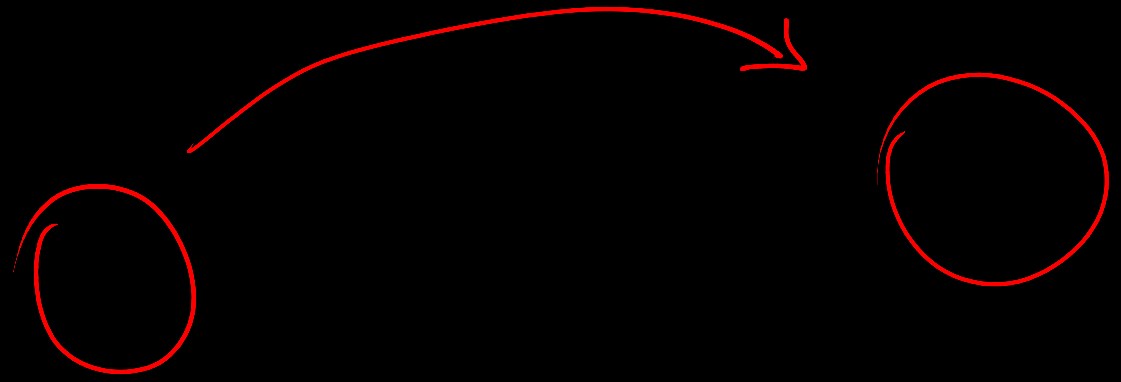

**Fig. 2.** The tracing property.

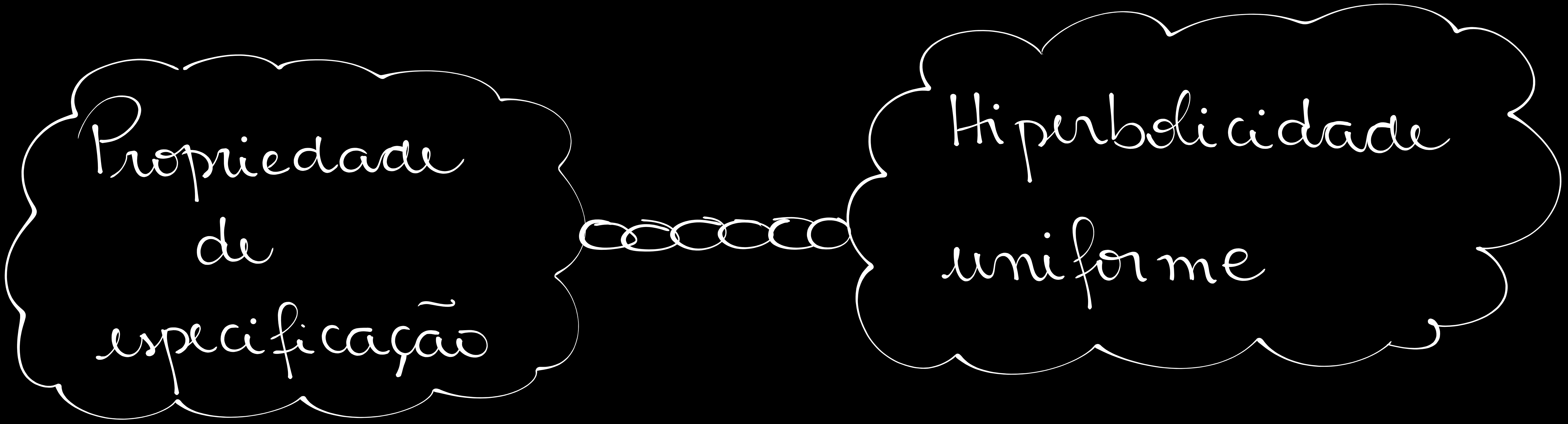
**Definition 2.2.** We say that $(X, f)$ has *the gluing orbit property*, if for every $\varepsilon > 0$ there is $M = M(\varepsilon) > 0$ such that for any orbit sequence $\mathscr{C}$, there is a gap $\mathscr{G} \in \Sigma_M$ such that $(\mathscr{C}, \mathscr{G})$ can be $\varepsilon$-traced.

são topologicamente transitivos.



transit. + shad $\Rightarrow$ gluing






Um difeo $C^1$-genérico com propriedade de especificação é um difeo Anosov transitivo.

Propriedade de sombreamento

Hiperbolicidade uniforme

conjectura

Um dife $C^1$-genérico com propriedade de sombreamento é Axioma A sem ciclos.

Specification-like properties Permitem Comportamentos não hiperbólicos

Exemplos de dinâmica satisfazendo a prop. gluing orbit:

1. Difeos hiperbólicos transitivos
2. Difeos parcialm. hiperbólicos obtidos como tempo 1 de fluxos Anosov [Bomfim, Varanda, 2019]
3. Equicontínuos minimais com entropia top. 0.

Ex: Considere a matriz

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
M = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

Grafo associado à $M$

$$
\begin{tikzcd}
1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \arrow[r, bend left] & 4 \arrow[l, bend left]
\end{tikzcd}
$$

Considere o conjunto das sequências admissíveis:

$$\Omega_M = \left\{ (x_k)_{k \in \mathbb{N}} \in \{1,2,3,4\}^{\mathbb{N}} \;\middle|\; \underbrace{M(x_k, x_{k+1})} = 1 \right\}$$

elemento da linha $x_k$ e coluna $x_{k+1}$ de $M$.

$$d((x_k),(y_k)) = 2^{-N}, \quad N = \min\{k \mid x_k \neq y_k\}$$

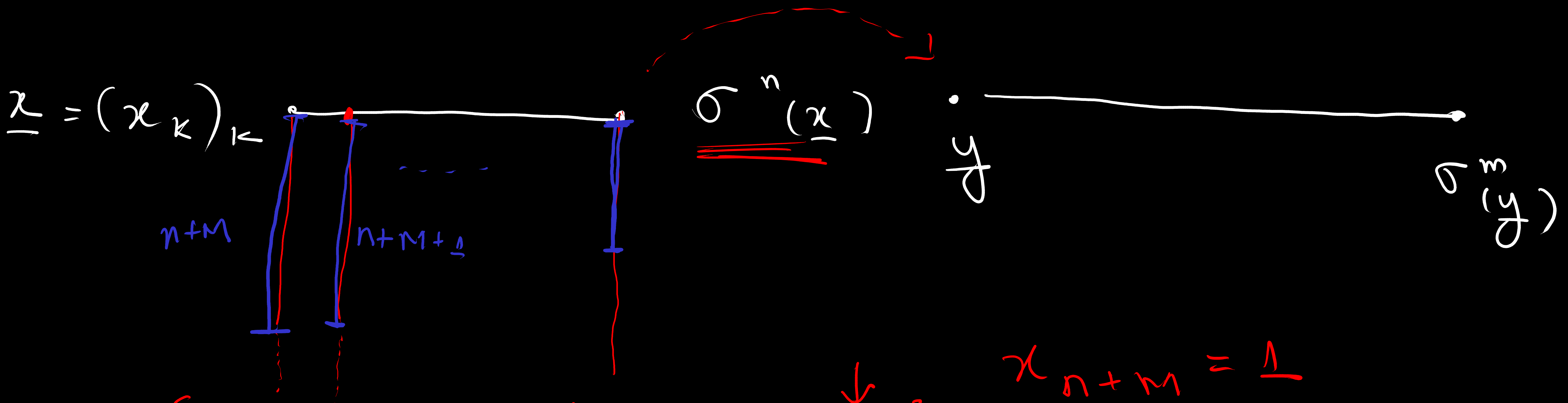
O shift $\sigma : \Omega_M \longrightarrow \Omega_M$ tem gluing orbit.

$$(x_k)_k \longmapsto (x_{k+1})_k$$

Obs: $\sigma$ não tem especificação.

$\forall \varepsilon > 0$ ¿$\exists\, M = M(\varepsilon)$?  Gap $\leq M + 2$

Considerar $M > 0$ i $2^{-M} < \varepsilon$




$x_{n+m} = 1$

$$z_k = \begin{cases} x_k, & k \in \{0 \to n+M\} \\ \left.\begin{matrix} 2 \\ 3 \end{matrix}\right\} \leftarrow & \\ y_k & \end{cases}$$

$z = (z_k) \in \Omega_E$

$y_0 = 4$

$\sigma$ ñ tem isp.

M, N pares

$\varepsilon > 0$ seja $M = \min\{n \in \mathbb{N} \mid 2^{-M} < \varepsilon\}$

$N > 0$ $M + N$

$\underline{x} = (1, 2, 1, 2, \ldots)$ $\qquad$ $y = (4, 3, 4, \ldots)$

$z$ sombreia $x$ e depois de $N+M$ passos sombreia $y$

$$z_k = \begin{cases} x_k, & k \in \{0, \ldots, M\} \\ \vdots & N \\ y_{k-(N+M)} & \end{cases}$$

$x_M = 1$

$\downarrow$ ímpar

$y_0 = 4$

Enquanto

$\boxed{\text{f com prop. especificação}} \Longrightarrow \boxed{h_{top}(f) > 0}$

...

$\boxed{\text{f com prop. gluing orbit pode ter } h_{top}(f) = 0}$

Caracterização...

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**Theorem 1.1.** *A topological dynamical system has both the gluing orbit property and zero topological entropy if and only if it is minimal and equicontinuous, i.e. it is topologically conjugate to a minimal rotation on a compact abelian group.*

Dado $\varepsilon > 0$

$\exists\ \delta > 0$ ;

$$B(x,\delta) \subset \cancel{W^{u}_{\varepsilon}}(x) \cap W^{s}_{\varepsilon}(x)$$

} $f$ é equicont.

$$
\begin{tikzcd}
(1) \arrow[r, shift left] \arrow[d] & (2) \arrow[l, shift left] & (5) \arrow[l] \\
(4) \arrow[r] & (3) \arrow[ul] \arrow[ur, "\text{with } (2)"'] &
\end{tikzcd}
$$

**Fig. 1.** Conditions in Theorem 1.2.

**Theorem 1.2.** *Let $(X, f)$ be a system with the gluing orbit property. The following are equivalent:*

*(1)* $(X, f)$ *has zero topological entropy.*
*(2)* $(X, f)$ *is minimal.*
*(3)* $(X, f)$ *is equicontinuous.*
*(4)* $(X, f)$ *is uniformly rigid.*
*(5)* $(X, f)$ *is uniquely ergodic.*

Quero mostrar que

$(1) \Rightarrow (4)$

Our proof of the relations between the conditions in Theorem 1.2 is illustrated in Fig. 1. It is well-known that (3) $\implies$ (1) and (2)(3) $\implies$ (5). We have shown in [31] that (1) $\implies$ (2) and (5) $\implies$ (2). We shall prove that (1) $\implies$ (4) in Section 3.2 and (4) $\implies$ (3) in Section 3.3. Finally, we prove that (2) $\implies$ (1) in Section 4.3.

Relação :

Propriedade gluing orbit $\longleftrightarrow$ Hiperbolicidade fraca

$\downarrow$

Exato p/ rotações minimais

Sabemos que :

$h_{top}(f) > 0 \Longrightarrow$ expoentes de Lyapunov não-nulos.

**Definition 2.9.** We say that $(X, f)$ is *equicontinuous* if the family $\{f^n\}_{n=0}^{\infty}$ is equicontinuous, i.e. for every $\varepsilon > 0$, there is $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$, we have

$$d(f^n(x), f^n(y)) < \varepsilon \text{ for every } n \in \mathbb{N}.$$

Definition.

We say that $(X, f)$ is *minimal* if $\mathrm{Tran}(X, f) = X$. Equivalently, there is no nonempty proper compact invariant subset of $X$.

$$\mathrm{Tran}(X, f) := \{x \in X : \overline{O(x)} = X\}$$

Artigo Sun

**Definition 2.16.** We say that $(X, f)$ is *uniformly rigid* if there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} D^0(f^{n_k}, \mathrm{Id}) = 0,$$

i.e. $f^{n_k} \to \mathrm{Id}$ uniformly.

Uniform rigidity is closely related to recurrence and almost periodicity.

$[0,1] \times S^1$

$\nexists$ unif rigido $\Rightarrow h_{top}(f) > 0$

**Lemma 3.2.** *Suppose that $(X, f)$ is not uniformly rigid. Then there is $\gamma > 0$ such that for every $p \in \mathrm{Tran}(X, f)$ and every $m \in \mathbb{Z}^+$, there is $\tau = \tau(p, m) \in \mathbb{N}$ such that*

$$d(f^\tau(p), f^\tau(f^m(p))) > \gamma.$$

$\exists\, \gamma > 0$ ;

$f^m(p) \notin W^{\gamma}_{\gamma}(p)$

**Lemma 3.2.** *Suppose that $(X, f)$ is not uniformly rigid. Then there is $\gamma > 0$ such that for every $p \in \mathrm{Tran}(X, f)$ and every $m \in \mathbb{Z}^+$, there is $\tau = \tau(p, m) \in \mathbb{N}$ such that*

$$d(f^\tau(p), f^\tau(f^m(p))) > \gamma.$$

Por absurdo, $\forall \frac{1}{2^k}$, $\exists$ $p \in \mathrm{Tran}(X,f)$ e $m_k \in \mathbb{N}$ ;

$$\underbrace{d(f^\tau(p), f^\tau(f^{m_k}(p)))}\leq \frac{1}{2^k} \quad \forall \tau$$

$$f^{n_\ell}(f^{m_k}(p))$$

Dado $x \in X$. $p \in \mathrm{Tran}(X,f)$.

$$x = \lim_\ell f^{n_\ell}(p) \Rightarrow f^{m_k}(x) = f^{m_k}(\lim_\ell f^{n_\ell}(p)) = \lim_\ell (f^{m_k}(f^{n_\ell}(p))) \quad \frac{1}{2^k}$$

$$\Rightarrow d(x, f^{m_k}(x)) \leq \frac{1}{2^k} \Rightarrow D^0(f^{m_k}, Id) \leq \frac{1}{2^k} \Rightarrow f^{m_k} \to Id. \quad \therefore$$

**Proposition 3.3.** *Suppose that $(X, f)$ has the gluing orbit property and it is not uniformly rigid. Then $h(f) > 0$.*

The rest of this subsection is devoted to the proof of Proposition 3.3.

Suppose that $(X, f)$ has the gluing orbit property and it is not uniformly rigid. Let $M := M(\varepsilon)$ be the constant in the gluing orbit property (as in Definition 2.2). By Proposition 2.13, $(X, f)$ is topologically transitive and hence $\mathrm{Tran}(X, f) \neq \emptyset$. By Lemma 3.2, there are $p \in \mathrm{Tran}(X, f)$, $\gamma > 0$ and $0 < \varepsilon < \frac{1}{3}\gamma$ such that for each $k = 1, 2, \cdots, 2M - 1$, there is $\tau_k \in \mathbb{N}$ such that

$$d(f^{\tau_k}(p), f^{\tau_k}(f^k(p))) > \gamma. \tag{2}$$

We fix

$$T := 2M + \max\{\tau_k : k = 1, \cdots, 2M - 1\}. \tag{3}$$

Note that $T$ depends exclusively on $p$ and $\varepsilon$. Denote

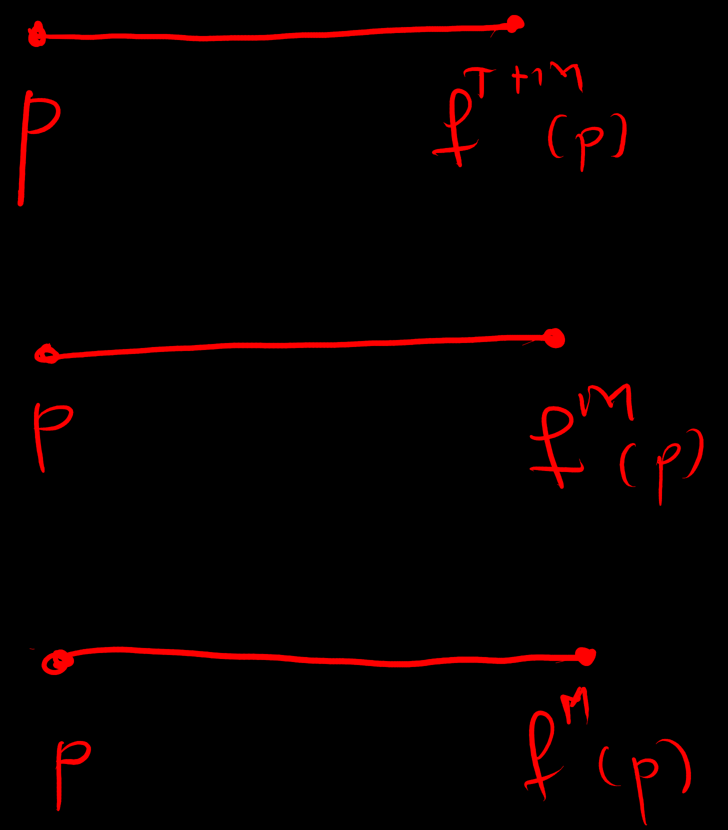
$$m_1 := T + M \text{ and } m_2 := T.$$

For each $\xi = \{\xi(k)\}_{k=1}^{\infty} \in \Sigma_2 := \{1, 2\}^{\mathbb{Z}^+}$, denote

$$\mathscr{C}_\xi := \{(p, m_{\xi(k)} + 1)\}_{k=1}^{\infty}.$$

$\xi \in \{T+M, M\}^{\mathbb{N}}$

$(T+M, M, M, T+M$



The gluing orbit property ensures that there are $z_\xi \in X$ and

$$\mathscr{G}_\xi = \{t_k(\xi)\}_{k=1}^{\infty} \in \Sigma_M$$

such that $(\mathscr{C}_\xi, \mathscr{G}_\xi)$ is $\varepsilon$-traced by $z_\xi$.

$2^N$

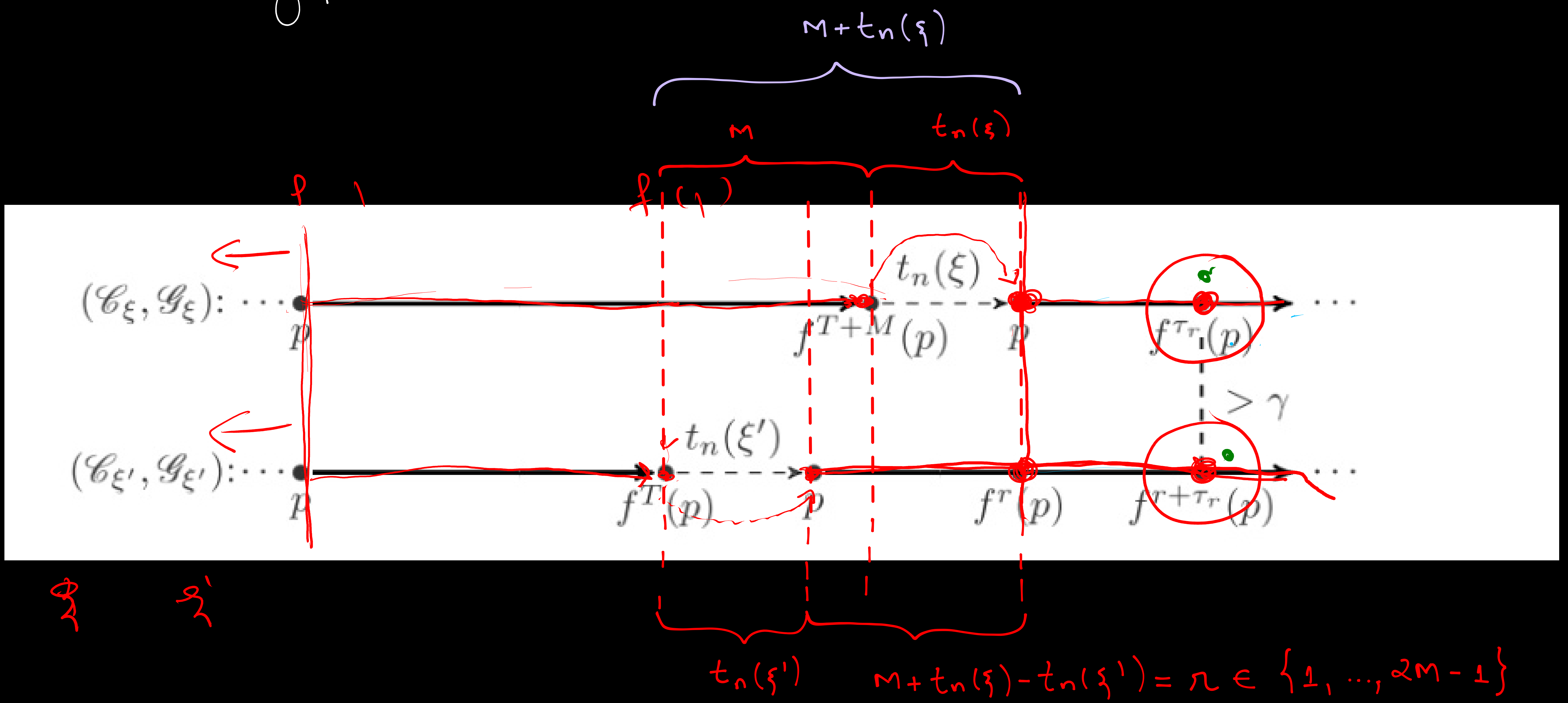
**Lemma 3.4.** *Let $N \geq 1$. If there is $n \in \{1, \cdots, N\}$ such that $\xi(n) \neq \xi'(n)$, then $z_\xi$ and $z_{\xi'}$ are $((N+1)(T+2M), \varepsilon)$-separated.*

$s(n,\delta)$ denota a maxima cardinalidade de um subconjunto $(n,\delta)$-separado de $X$.

$$h(f,\delta) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f,\delta) \qquad h(f) = \lim_{\delta\to 0} h(f,\delta).$$

$$\geq \limsup \frac{1}{(N+1)(T+2M)} \log 2^N > 0$$

Caso 1: Os tamanhos dos pedaços das órbitas de p das seq. se diferem antes dos gaps.



Caso 2: Os gaps se diferem antes dos pedaços de órbitas.

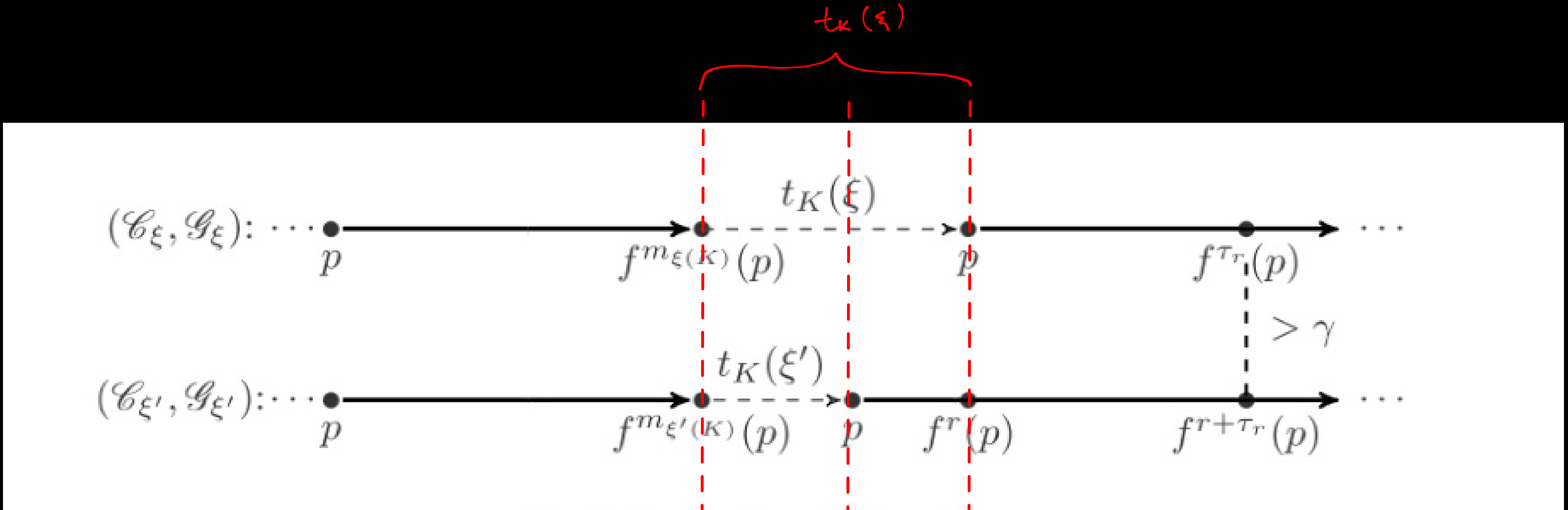



**Fig. 4.** Separation in Case 2.

$t_K(\xi')$

$t_K(\xi) - t_K(\xi') = r \in \{1, \dots, 2M-1\}$.

# Dicotomia

Se $f: X \to X$ tem propriedade gluing orbit, então $f$ é equicontínuo ou $f$ é sensível.

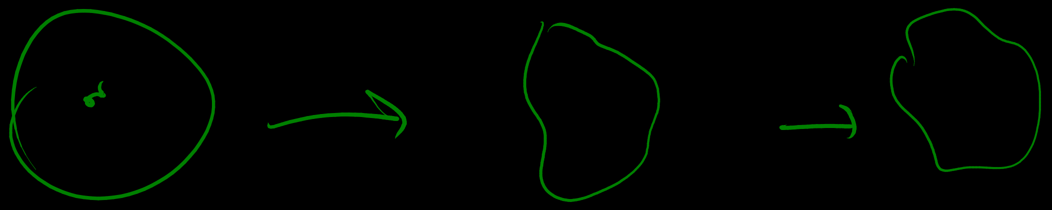
- $h_{top}(f) = 0 \iff f$ é equicontínuo
- $h_{top}(f) > 0 \iff f$ é sensível

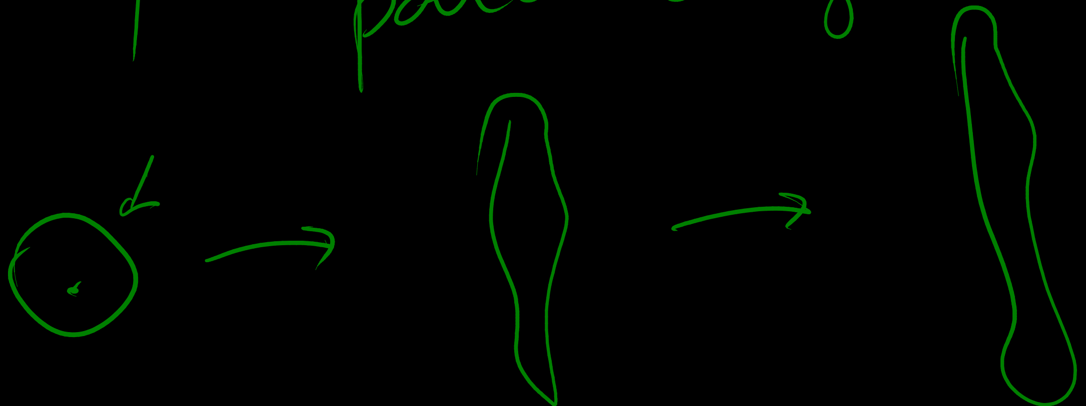
$\exists\, \varepsilon > 0$ ; dado $x \in X$ $\forall\, \delta > 0$

$\exists\, y \in B(x, \delta)$ tq

$d(f^n(x), f^n(y)) > \varepsilon$

para algum $n \in \mathbb{N}$





**Prop**: Se $f: X \to X$ é transitivo e $\underbrace{\text{não é uniformemente rígido}}$ então $f$ é sensível.

$\Updownarrow$ g.o

$h_{top}(f) > 0$

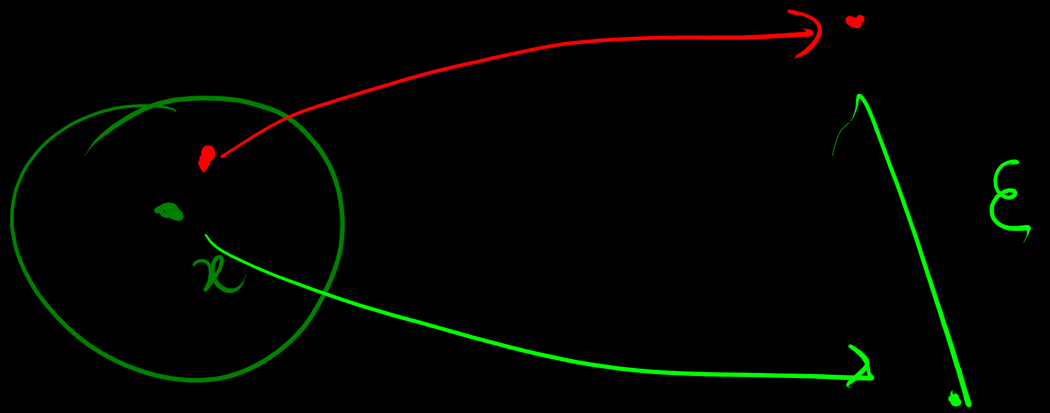
Existe $\gamma > 0$ do Lema 3.2

$\vdash$: $\varepsilon = \frac{\gamma}{4}$ é a constante de sensibilidade de $f$.

**Lemma 3.2.** *Suppose that $(X, f)$ is not uniformly rigid. Then there is $\gamma > 0$ such that for every $p \in \mathrm{Tran}(X, f)$ and every $m \in \mathbb{Z}^+$, there is $\tau = \tau(p, m) \in \mathbb{N}$ such that*

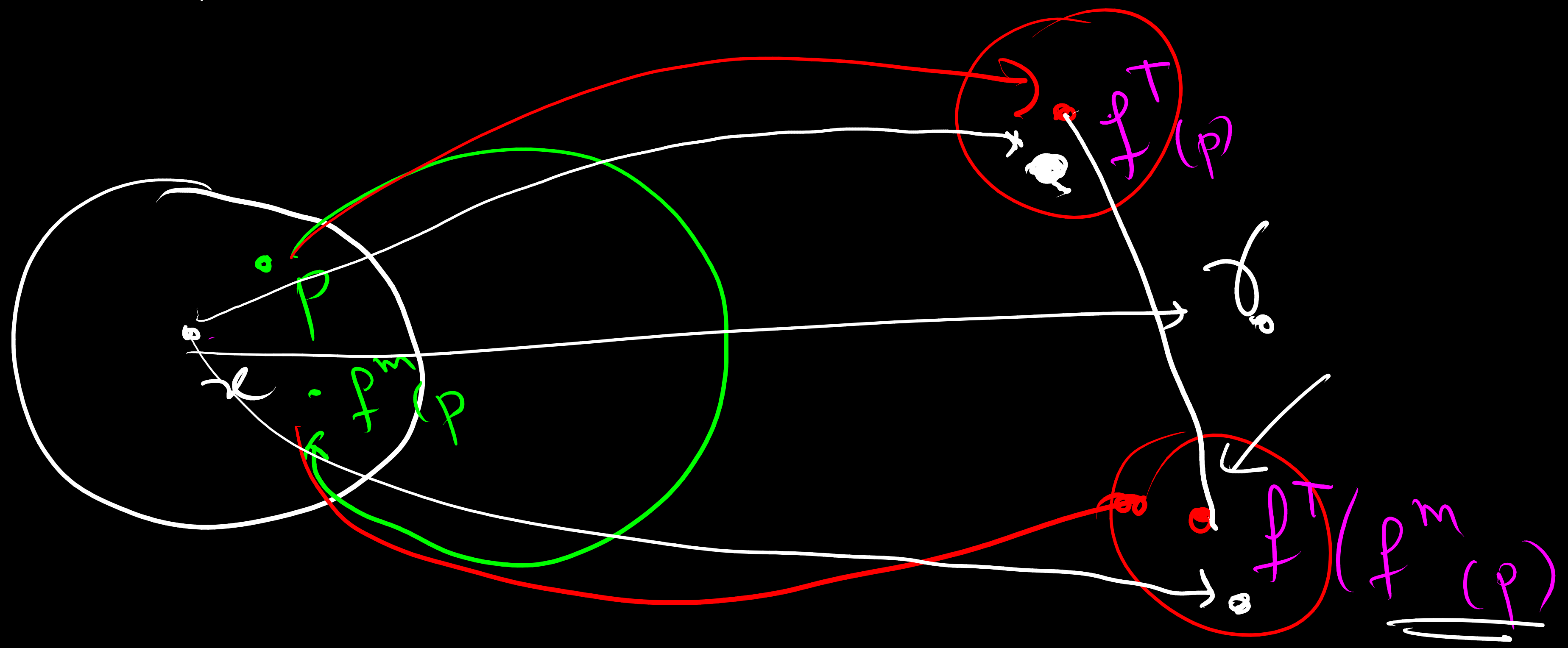
$$d(f^\tau(p), f^\tau(f^m(p))) > \gamma.$$

$f^m(p) \notin W^\tau_\gamma(p)$ $\forall m \in \mathbb{N}$.




$$\varepsilon = \gamma/4$$

$$p \in \mathrm{Tran}(X, f)$$

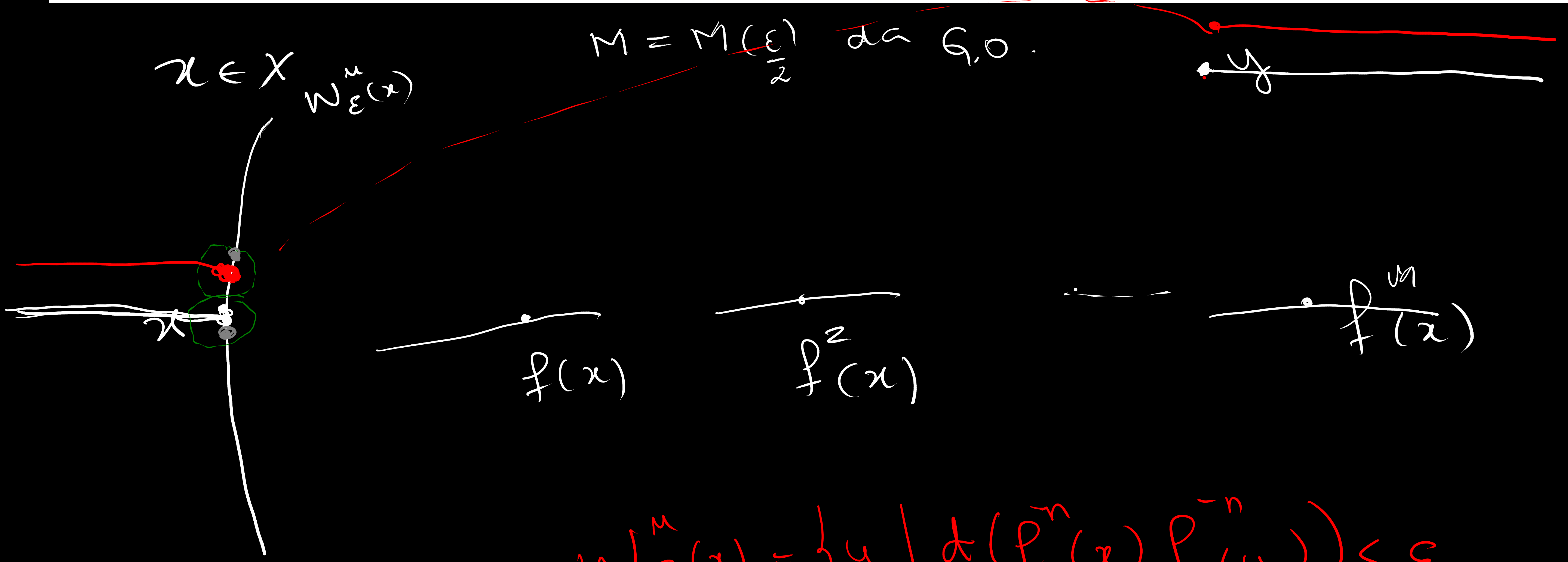


—, Carvalho, Cordeiro (2024)

**Theorem A.** *If $f\colon X \to X$ is a homeomorphism of an infinite compact metric space $X$ satisfying the gluing-orbit property, then there is $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and each $x \in X$ there are compact and perfect sets*

$$C_x \subset W^s_\varepsilon(x) \quad \text{and} \quad D_x \subset W^u_\varepsilon(x).$$

$M = M(\frac{\varepsilon}{2})$ da G.O.




$$W^u_\varepsilon(x) = \{y \mid d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon$$